Introduction to (some aspects of) linear dynamics

ERNST Romuald

Université Blaise-Pascal de Clermont-Ferrand
Workshop, University of Málaga

12th March, 2013

Contents

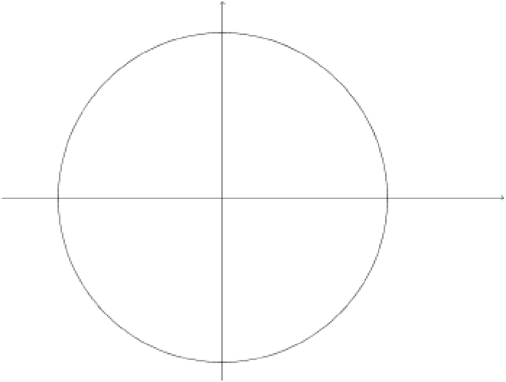
Definition

A linear operator T is said to be hypercyclic if there exists a vector $x \in X$ such that its orbit by T is dense in X.

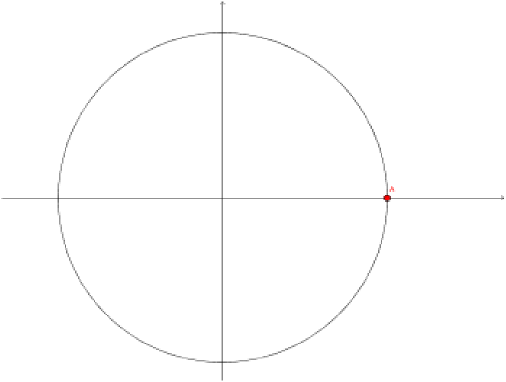
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:

A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:

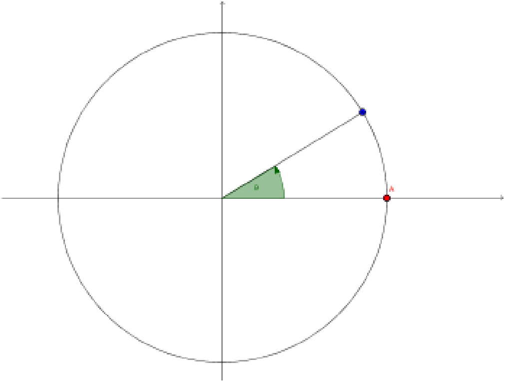
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



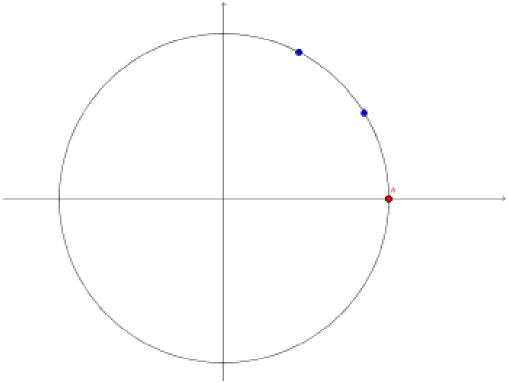
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



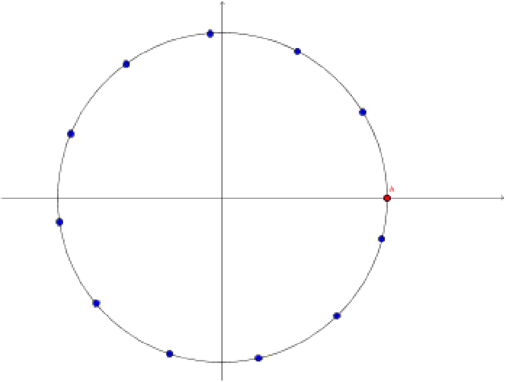
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



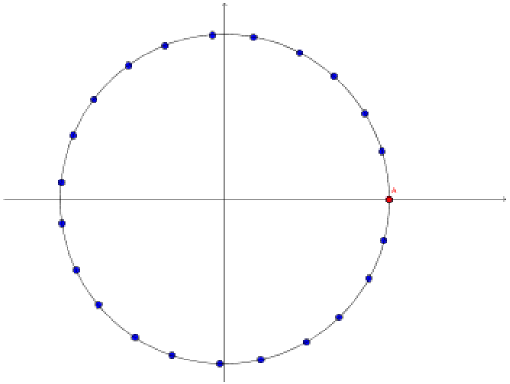
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



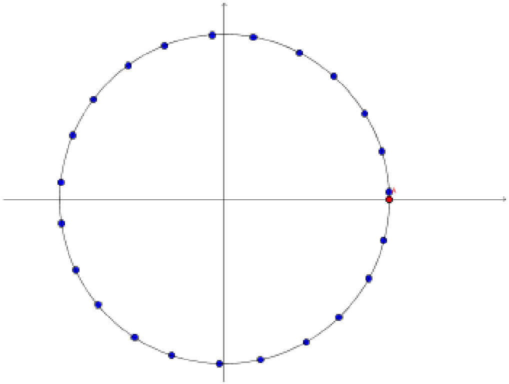
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



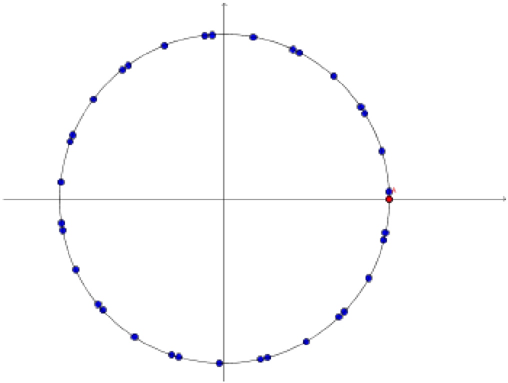
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



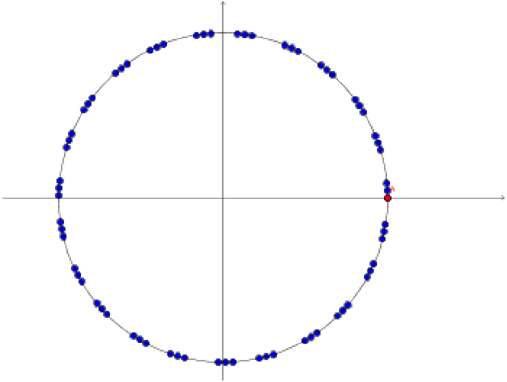
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



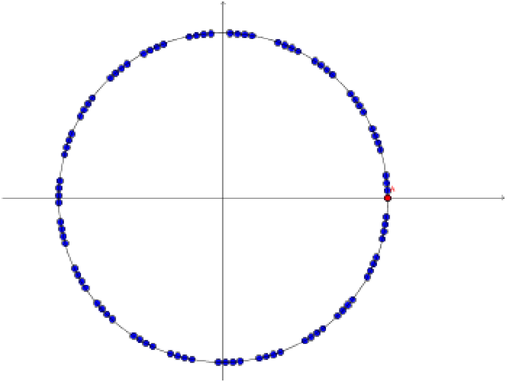
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



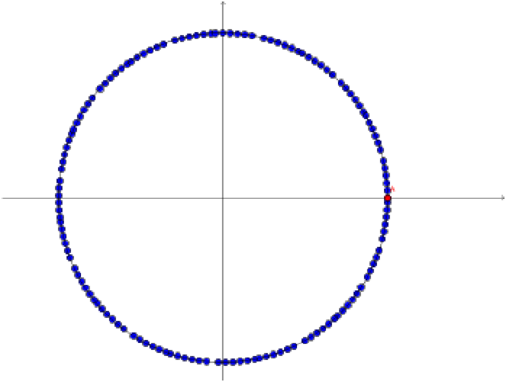
A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



A "hypercyclic" operator on the unit circle in $\mathbb{R}^2$:



Definition

A linear operator T is said to be supercyclic if there exists a vector $x \in X$ such that $\overline{\{T^n(\mathbb{K}.x), n \in \mathbb{N}\}} = X$.

hypercyclicity

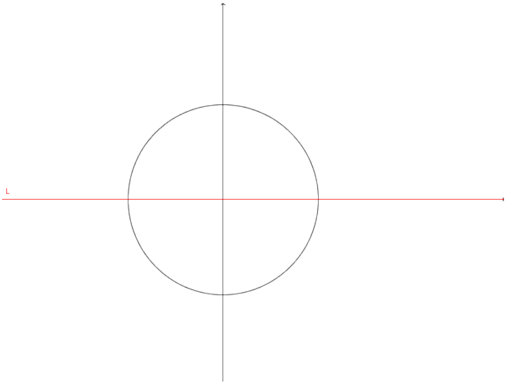
$$\Downarrow$$

supercyclicity

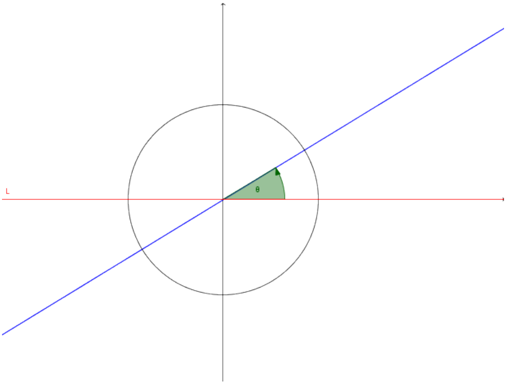
A supercyclic rotation in $\mathbb{R}^2$:

A supercyclic rotation in $\mathbb{R}^2$:

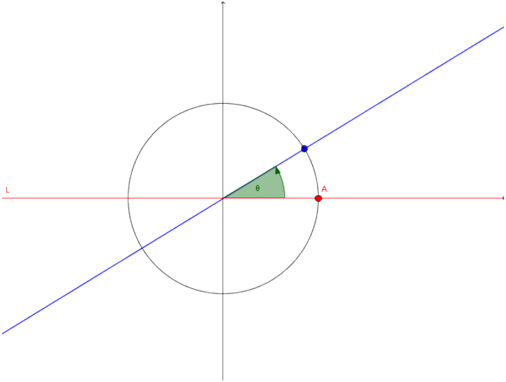
A supercyclic rotation in $\mathbb{R}^2$:



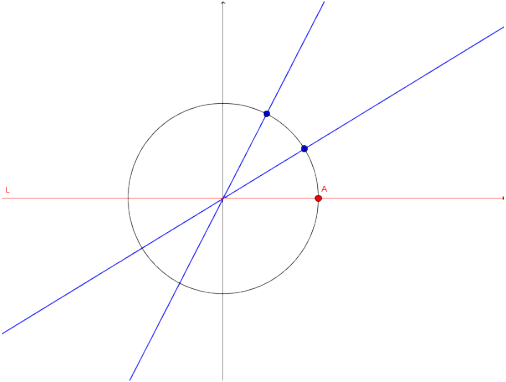
A supercyclic rotation in $\mathbb{R}^2$:



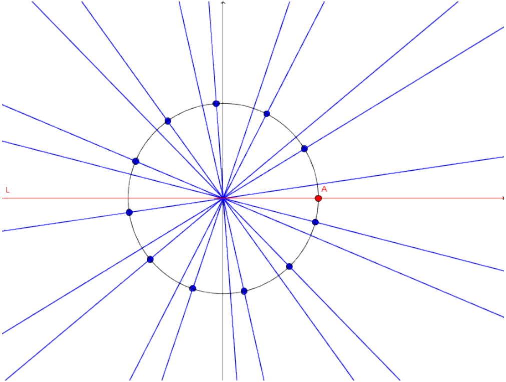
A supercyclic rotation in $\mathbb{R}^2$:



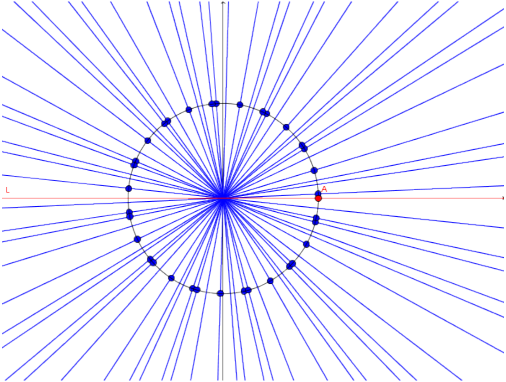
A supercyclic rotation in $\mathbb{R}^2$:



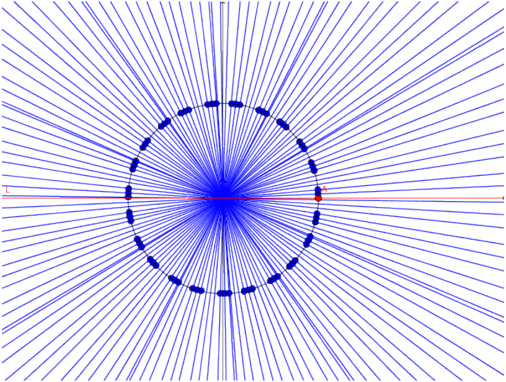
A supercyclic rotation in $\mathbb{R}^2$:



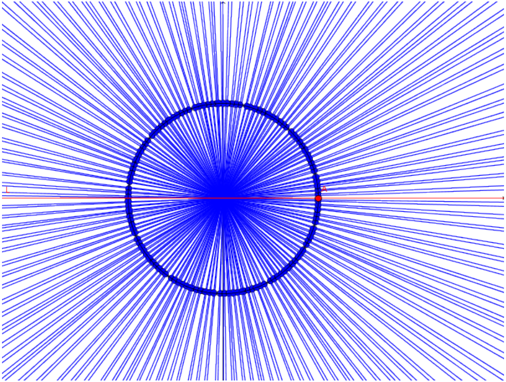
A supercyclic rotation in $\mathbb{R}^2$:



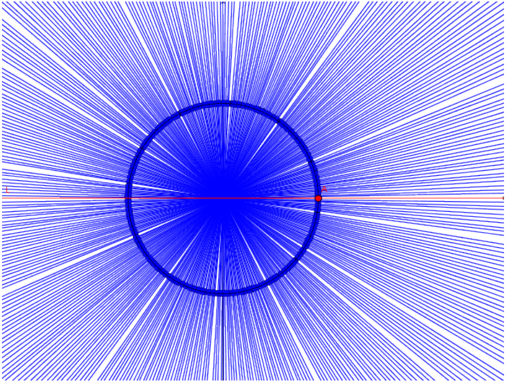
A supercyclic rotation in $\mathbb{R}^2$:



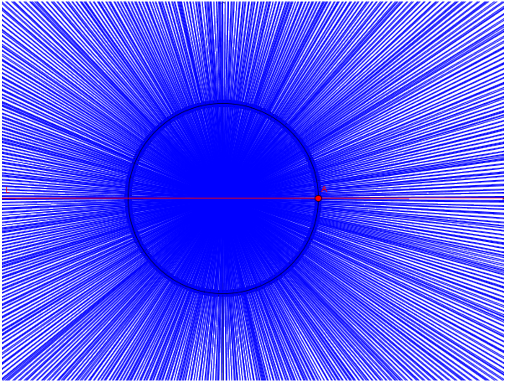
A supercyclic rotation in $\mathbb{R}^2$:



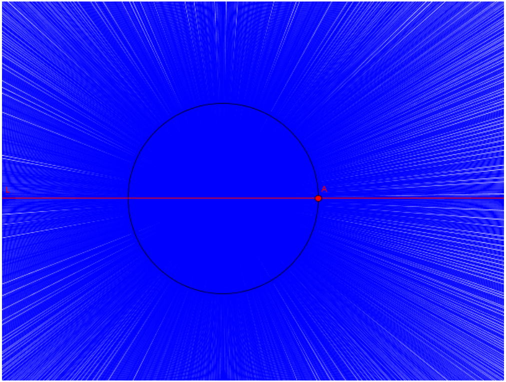
A supercyclic rotation in $\mathbb{R}^2$:



A supercyclic rotation in $\mathbb{R}^2$:



A supercyclic rotation in $\mathbb{R}^2$:



Proposition

Let X be a separable Banach space and T a hypercyclic operator on X. Let $(U_i)_{i\in\mathbb{N}}$ be a countable basis of open sets. Then the set of hypercyclic vectors for T can be described as follows :

$$HC(T) = \bigcap_{i\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} T^{-n}(U_i).$$

In particular, $HC(T)$ is a dense G_δ-set of X.

Proposition

Let X be a separable Banach space and T a supercyclic operator on X. Let $(U_i)_{i\in\mathbb{N}}$ be a countable basis of open sets. Then the set of supercyclic vectors for T can be described as follows :

$$SC(T) = \bigcap_{i\in\mathbb{N}} \bigcup_{\substack{n\in\mathbb{N}\\ \lambda\in\mathbb{K}}} (\lambda T^n)^{-1}(U_i).$$

In particular, $SC(T)$ is a dense G_δ-set of X.

Moreover, Ansari proved the following result :

Proposition

Let k be a positive integer. Then T is hypercyclic if and only if T^k is hypercyclic.

Moreover, Ansari proved the following result :

Proposition

Let k be a positive integer. Then T is supercyclic if and only if T^k is supercyclic.

Contents

Definition (Feldman, 2002)

A linear operator T is said to be *n-supercyclic*, $n \geq 1$, if there exists a linear subspace of X with dimension n such that its orbit by T is dense in X.

hypercyclicity

$$\Downarrow$$

supercyclicity

$$hypercyclicity$$
$$\Downarrow$$
$$supercyclicity$$
$$\Downarrow$$
$$n - supercyclicity$$

hypercyclicity

$\Downarrow$

supercyclicity

$\Downarrow$

$2 - supercyclicity$

$\Downarrow$

$\vdots$

$\Downarrow$

$n - supercyclicity$

A 2-supercyclic rotation in $\mathbb{R}^3$:

Question

Is the set of n-supercyclic subspaces a dense G_δ-set ? ? ?

Question

Is the set of n-supercyclic subspaces a dense G_δ-set ? ? ?

- In which space ?

Open Question

Let k be a non-negative integer. Is that true that T is n-supercyclic if and only if T^k is n-supercyclic ???

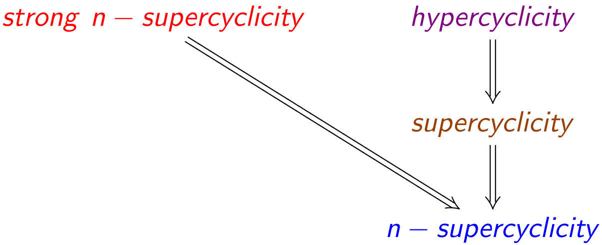
Definition (Feldman, 2002)

A linear operator T is said to be n-supercyclic, $n \geq 1$, if there exists a linear subspace of X with dimension n such that its orbit by T is dense in X.

Definition (Shkarin, 2008)

A linear operator T is said to be strongly n-supercyclic, $n \geq 1$, if there exists a linear subspace L of X with dimension n such that for every $k \in \mathbb{N}$, the subspace $T^k(L)$ has dimension n and the set $\left\{T^k(L)\right\}_{k\in\mathbb{N}}$ is a dense subset of $\mathbb{P}_n(X)$.

$$\textit{hypercyclicity}$$
$$\Downarrow$$
$$\textit{supercyclicity}$$
$$\Downarrow$$
$$n - \textit{supercyclicity}$$


strong n − supercyclicity
hypercyclicity
supercyclicity
n − supercyclicity

Proposition

The following assertions are equivalent :

(i) *T is strongly n-supercyclic ;*

(ii) *There exists a subspace L with dimension n such that ($T^k(L)$ has dimension n for all k) :*
$\cup_{i=1}^{\infty} T^i(L) \times \cdots \times T^i(L)$ *is dense in* X^n.

Proposition

The following assertions are equivalent :

(i) T *is strongly n-supercyclic ;*

(ii) *There exists a subspace* L *with dimension* n *such that* $(T^k(L)$ *has dimension* n *for all* $k)$ *:* $\cup_{i=1}^{\infty} T^i(L) \times \cdots \times T^i(L)$ *is dense in* X^n.

(iii) $\forall U \subset \mathbb{P}_n(X), \forall V \subset X^n$ *non-empty open sets,* $\exists i \in \mathbb{N} : \ (\oplus_{k=1}^{n} T)^i(\pi_n^{-1}(U)) \cap V \neq \emptyset$.

Moreover, $\mathcal{E}S_n(T)$ *is a dense* G_δ *set of* $\mathbb{P}_n(X)$.

Theorem (Shkarin, 2008)

Let $k, n \in \mathbb{N}^*$. Then T is strongly n-supercyclic if and only if T^k is strongly n-supercyclic.

Contents

Theorem (Bourdon,Feldman,Shapiro, 2004)

Let $n \geq 2$. There is no k-supercyclic operator on $\mathbb{C}^n$ with $1 \leq k < n$.

Theorem (Bourdon,Feldman,Shapiro, 2004)

Let $n \geq 2$. There is no k-supercyclic operator on $\mathbb{C}^n$ with $1 \leq k < n$.

Theorem

Let $n \geq 3$. There is no strongly k-supercyclic operator on $\mathbb{R}^n$ with $1 \leq k < n$.

Contents

Corollary

Let S be an operator satisfying the Hypercyclicity Criterion on a Banach space Y. Then, $T = Id \oplus S$ is a strongly k-supercyclic operator on $X = \mathbb{K}^n \oplus Y$ if and only if $k \geq n$.

Theorem

Let X be a Banach space which admits a normalized unconditional basis $(e_i)_{i\in\mathbb{N}}$ for which the forward shift operator is continuous. Then, there exists a supercyclic operator which is not strongly p-supercyclic for any $p \geq 2$.

Contents

Theorem (Feldman, 2002, The Circle Theorem)

If T is a n-supercyclic operator, then there exists n circles $\Gamma_i = \{z : |z| = r_i\}$, $r_i \geq 0$, $i = 1, \ldots, n$, such that every connected component of the spectrum of T intersects $\cup_{i=1}^n \Gamma_i$.

Theorem (Feldman, 2002, The Circle Theorem)

If T is a n-supercyclic operator, then there exists n circles $\Gamma_i = \{z : |z| = r_i\}$, $r_i \geq 0$, $i = 1, \ldots, n$, such that every connected component of the spectrum of T intersects $\cup_{i=1}^{n} \Gamma_i$.

Theorem

Let T be a strongly n-supercyclic operator on a Banach space X. Then, there exists two linear subspaces, F with dimension $\leq n$ and X_0, invariant for T such that $X = F \oplus X_0$. There exists also $R \geq 0$ such that every connected component of the spectrum of $T_0 := T_{|X_0}$ intersects the circle $\{z \in \mathbb{C} : |z| = R\}$.

Muchas Gracias !