NON-EXISTENCE OF FREQUENTLY HYPERCYCLIC SUBSPACES FOR P(D)

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Abstract. We answer a question posed by Bonilla and Grosse-Erdmann by showing that the operators $P(D)$ on the space of entire functions $H(\mathbb{C})$, where D is the differentiation operator and P is a polynomial, do not possess a frequently hypercyclic subspace.

1. Introduction

Let X be a separable infinite-dimensional Fréchet space and T a continuous linear operator on X.

We are interested in the properties of the orbits of T. For instance, we say that T is *hypercyclic* if there exists a vector $x \in X$ (also said hypercyclic) such that $\{T^n x : n \geq 0\}$ is dense in X. In other words, T is hypercyclic if there exists a vector $x \in X$ such that for any non-empty open set U in X the set $N(x, U) := \{n \geq 0 : T^n x \in U\}$ is non-empty, or equivalently infinite since X does not contain isolated points. Two recent books by Bayart and Matheron [4] and Grosse-Erdmann and Peris [15] have been written on this topic and the reader can refer to these for more details on the notions involved in this article.

The first examples of hypercyclic operators are the non-trivial translation operators [8] and the differentiation operator D [18] on the space of entire functions $H(\mathbb{C})$. Obviously, each of these operators commute with the differentiation operator D. It was in fact shown that every operator, that commutes with D and is not a scalar multiple of the identity, is hypercyclic [13].

Given a hypercyclic operator T, one can wonder which is the structure of the set of hypercyclic vectors for T. At the beginning of the twentieth century, Birkhoff [7] showed that if T is hypercyclic then T possesses a dense G_δ set of hypercyclic vectors. On the other hand, although the sum of two hypercyclic vectors may not be hypercyclic, Bourdon [12] and Herrero [16] showed that if T is hypercyclic then there exists a dense infinite-dimensional subspace in which each non-zero vector is hypercyclic.

In 1995, Bernal and Montes [5] then remarked that each non-trivial translation operator on $H(\mathbb{C})$ possesses an infinite-dimensional closed subspace

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in which each non-zero vector is hypercyclic. Such an infinite-dimensional closed subspace is called a *hypercyclic subspace*. Several criteria for the existence and the non-existence of hypercyclic subspaces were found [23, 17, 9, 24, 19, 20] and a characterization of the existence of hypercyclic subspaces was even obtained in the case of complex Banach spaces [14]. On the Fréchet space $H(\mathbb{C})$, we know that each operator, that commutes with D and is not a scalar multiple of the identity, possesses a hypercyclic subspace. Indeed, every operator commuting with D is of the form $\varphi(D)$ where φ is an entire function of exponential type [13] and the existence of a hypercyclic subspace was obtained by Petersson [24] for the operators $\varphi(D)$ where φ is an entire function of exponential type that is not a polynomial, by Shkarin [26] for the operator D and by Menet [19] for the operators $P(D)$ where P is a non-constant polynomial.

In a different direction, if x is a hypercyclic vector for T, one can wonder which "size" the infinite sets $N(x,U)$ can have. In particular, one can wonder if these sets can have a positive lower density or a positive upper density. This idea has been used by Bayart and Grivaux [1] to define the notion of frequent hypercyclicity and by Shkarin [25] to define the notion of $\mathcal{U}$-frequent hypercyclicity.

Definition 1.1. An operator $T \in L(X)$ is said to be *frequently hypercyclic* (resp. *$\mathcal{U}$-frequently hypercyclic*) if there exists a vector $x \in X$ such that for any non-empty open set U in X we have $\underline{\mathrm{dens}}(N(x,U)) > 0$ (resp. $\overline{\mathrm{dens}}(N(x,U)) > 0$). Such a vector x is said to be frequently hypercyclic (resp. $\mathcal{U}$-frequently hypercyclic).

We recall that if A is a set of non-negative integers then the lower density and the upper density of A are respectively given by

$$\underline{\mathrm{dens}}(A) := \liminf_{k\to\infty} \frac{\#\{n \le k : n \in A\}}{k} \quad \text{and} \quad \overline{\mathrm{dens}}(A) := \limsup_{k\to\infty} \frac{\#\{n \le k : n \in A\}}{k}.$$

The non-trivial translation operators and the differentiation operator D are examples of frequently hypercyclic operators [3]. We can even show that every operator, that commutes with D and is not a scalar multiple of the identity, is frequently hypercyclic [10]. In fact, these operators were already known to be chaotic [13] and the comprehension of the links between chaotic and frequently hypercyclic operators has been an important problem in the last decade. Bayart and Grivaux [2] gave a first answer to this question showing that there exists a weighted shift on c_0 which is frequently hypercyclic but not chaotic. Conversely, Menet [22] has constructed a chaotic operator which is not $\mathcal{U}$-frequently hypercyclic, hence not frequently hypercyclic proving that, in the general setting no link can exist between linear chaos and frequent hypercyclicity.

Given a frequently hypercyclic operator T, one can wonder if T possesses a frequently hypercyclic subspace i.e. an infinite-dimensional closed subspace in which each non-zero vector is frequently hypercyclic. This notion was investigated for the first time by Bonilla and Grosse-Erdmann [11]. In particular, they proved that if φ is an entire function of exponential type that is not a polynomial then $\varphi(D)$ possesses a frequently hypercyclic subspace and they posed the following question:

Question 1 ([11, Problem 2])**.** Does the differentiation operator D on $H(\mathbb{C})$ have a frequently hypercyclic subspace?

This question can be naturally generalized as follows:

Question 2. Do the operators $P(D)$ on $H(\mathbb{C})$, where P is a non-constant polynomial, have a frequently hypercyclic subspace?

The answer to these questions is not obvious since the operators $P(D)$, where P is a non-constant polynomial, are frequently hypercyclic and possess a hypercyclic subspace. The goal of this paper consists in proving the following theorem that answers negatively Question 1 and Question 2.

Theorem 1.2. *Let D be the differentiation operator on $H(\mathbb{C})$ and P be a polynomial. Then $P(D)$ does not possess any frequently hypercyclic subspace.*

In their paper [11], Bonilla and Grosse-Erdmann also posed the following question:

Question 3 ([11, Problem 1])**.** Does there exist a frequently hypercyclic operator that has a hypercyclic subspace but not a frequently hypercyclic subspace?

The answer to this question was given by Menet [21], who found a frequently hypercyclic weighted shift on l^p possessing a hypercyclic subspace and no frequently hypercyclic subspace. We remark that thanks to Theorem 1.2 we now know a simple family of operators having this property. In fact, it is shown in [6] that the operators $P(D)$, where P is a non-constant polynomial, even possess a $\mathcal{U}$-frequently hypercyclic subspace i.e. an infinite-dimensional closed subspace in which each non-zero vector is $\mathcal{U}$-frequently hypercyclic. The operators $P(D)$, where P is a non-constant polynomial, thus form a family of frequently hypercyclic operators possessing a $\mathcal{U}$-frequently hypercyclic subspace and no frequently hypercyclic subspace.

We divide the proof of the non-existence of frequently hypercyclic subspaces for the operators $P(D)$ in two sections. In Section 2, we give a general criterion for an operator on a Fréchet space ensuring the non-existence of frequently hypercyclic subspaces. Then, we restrict ourselves to study under which conditions a weighted shift on a Köthe sequence space does not possess a frequently hypercyclic subspace, allowing a much simpler expression of the previous criterion and we show that the differentiation operator satisfies these conditions. In Section 3, we show how to adapt our arguments in order to prove the non-existence of frequently hypercyclic subspaces for every operator of the form $P(D)$ where P is a polynomial. The proof of this result follows the same lines as for the differentiation operator but technical issues force us to use directly the general criterion.

2. Case of weighted shifts and the operator D

We start by generalizing to Fréchet spaces the sufficient condition for the non-existence of frequently hypercyclic subspaces given in [21].

Theorem 2.1. *Let X be a separable infinite-dimensional Fréchet space, (p_j) an increasing sequence of seminorms inducing the topology of X and $T \in L(X)$. If for any infinite-dimensional closed subspace M of X, there exist $C > 0$ and $j_0 \geq 1$ such that for any $K \geq 1$, any $x \in X$, any $j \geq 1$, there exists $x' \in M$ such that*

$$p_j(x') \leq \frac{1}{K} \quad \textit{and} \quad \sup_{k>K} \frac{\#\{n \leq k : p_{j_0}(T^n(x+x')) \geq C\}}{k} \geq 1 - \frac{1}{K},$$

then T does not possess any frequently hypercyclic subspace.

Proof. Let M be an infinite-dimensional closed subspace of X. Let $C > 0$ and $j_0 \geq 1$ such that for any $K \geq 1$, any $x \in X$, any $j \geq 1$, there exists $x' \in M$ such that

$$p_j(x') \leq \frac{1}{K} \quad \text{and} \quad \sup_{k>K} \frac{\#\{n \leq k : p_{j_0}(T^n(x+x')) \geq C\}}{k} \geq 1 - \frac{1}{K}.$$

We show that there exists a vector $x \in M$ such that

$$\overline{\text{dens}}\Big\{n \geq 0 : p_{j_0}(T^n x) \geq \frac{C}{2}\Big\} := \limsup_{k\to\infty} \frac{\#\big\{n \leq k : p_{j_0}(T^n x) \geq \frac{C}{2}\big\}}{k} = 1.$$

This will imply that M is not a frequently hypercyclic subspace since we would have, for some $x \in M\backslash\{0\}$,

$$\underline{\text{dens}}\ N\Big(x, \Big\{y \in X : p_{j_0}(y) < \frac{C}{2}\Big\}\Big) = 1 - \overline{\text{dens}}\Big\{n \geq 0 : p_{j_0}(T^n x) \geq \frac{C}{2}\Big\} = 0.$$

We warn the reader here, that one cannot expect a similar result above with an upper density instead of the lower density. Indeed, this theorem is used later to exhibit operators having $\mathcal{U}$-frequently hypercyclic subspaces but no frequently hypercyclic subspaces. This prevents us from making the following computations in a rough manner.

For this, we construct a sequence $(x_j)_{j\geq 0} \subset M$ with $x_0 = 0$ and an increasing sequence $(k_j)_{j\geq 0}$ with $k_0 = 0$ such that for any $j \geq 1$,

(1) $p_j(x_j) \leq \frac{1}{j^2}$;
(2) for any $n \leq k_{j-1}$, $p_{j_0}(T^n x_j) < \frac{C}{2^j}$;
(3) we have

$$\frac{\#\{n \leq k_j : p_{j_0}(T^n(\sum_{l=1}^{j} x_l)) \geq C\}}{k_j} \geq 1 - \frac{1}{j}.$$

By continuity, we know that for any $j \geq 1$, there exists $m_j \geq j$ and $C_j > 0$ such that for any $x \in X$, any $n \leq k_{j-1}$,

$$p_{j_0}(T^n x) \leq C_j p_{m_j}(x).$$

At each step, we obtain therefore a vector x_j satisfying the desired properties for some $k_j > k_{j-1}$ by using our assumption for $K > \max\{j^2, k_{j-1}, \frac{C_j 2^j}{C}\}$, $x = \sum_{l=1}^{j-1} x_l$ and the seminorm p_{m_j}.

Let $I_j = \{n \le k_j : p_{j_0}(T^n(\sum_{l=1}^{j} x_l)) \ge C\}$ and $x := \sum_{j=1}^{\infty} x_j$, which converges by (1). Since M is closed, we know that $x \in M$ and for any $j \ge 1$, any $n \in I_j$, we have

$$\begin{aligned} p_{j_0}(T^n x) &\ge p_{j_0}\Big(\sum_{l=1}^{j} T^n x_l\Big) - \sum_{l=j+1}^{\infty} p_{j_0}(T^n x_l) \\ &\ge C - \sum_{l=j+1}^{\infty} \frac{C}{2^l} \ge \frac{C}{2} \qquad \text{by definition of } I_j \text{ and (2).} \end{aligned}$$

We deduce that for any $j \ge 1$,

$$\frac{\#\{n \le k_j : p_{j_0}(T^n x) \ge \frac{C}{2}\}}{k_j} \ge \frac{\# I_j}{k_j} \ge 1 - \frac{1}{j} \xrightarrow[j\to\infty]{} 1.$$

The result follows. $\square$

When we will apply Theorem 2.1 to weighted shifts on Köthe sequence spaces, we will simplify one of the two conditions of this theorem by observing that if B_w is a weighted shift on a Köthe sequence space then for any $k, n \ge 0$,

$$p_j(B_w^n(x + x')) \ge p_j(B_w^n((x_k + x'_k)e_k)).$$

We recall that a weighted shift B_w is a map defined by

$$B_w(x_0, x_1, x_2, \cdots) = (w_1 x_1, w_2 x_2, \cdots)$$

where $w = (w_n)_{n\ge 1}$ is a sequence of non-zero scalars and that the Köthe sequence spaces are defined as follows:

Definition 2.2. Let $A^+ = (a_{j,k})_{j\ge 1, k\ge 0}$ be a matrix such that for any $j \ge 1$, any $k \ge 0$, we have $a_{j,k} > 0$ and $a_{j,k} \le a_{j+1,k}$. The (real or complex) *Köthe sequence spaces* $\lambda^p(A^+)$ and $c_0(A^+)$ are defined by

$$\lambda^p(A^+) := \Big\{(x_k) \in \mathbb{K}^{\mathbb{N}} : p_j((x_k)) = \Big(\sum_{k=0}^{\infty} |x_k a_{j,k}|^p\Big)^{\frac{1}{p}} < \infty,\ j \ge 1\Big\} \ \text{ and}$$

$$c_0(A^+) := \{(x_k) \in \mathbb{K}^{\mathbb{N}} : \lim_{k\to\infty} |x_k| a_{j,k} = 0,\ j \ge 1\} \text{ with } p_j((x_k)_k) = \sup_{k\ge 0} |x_k| a_{j,k}.$$

The following lemma will be particularly useful for applying Theorem 2.1 to weighted shifts on the Köthe sequence spaces and for showing that the operators of the form $P(D)$ do not possess a frequently hypercyclic subspace. We recall that, if $x = (x_k)_{k\ge 0}$ is a sequence of complex numbers, its valuation $v(x)$ is defined by

$$v(x) := \inf\{k \ge 0 : x_k \ne 0\}.$$

Lemma 2.3. *Let $X = \lambda^p(A^+)$ and M be an homogeneous set of X containing non-zero vectors with arbitrary large valuation. Then for any $K \geq 1$, any $j \geq 1$, any $x, f \in X$, there exists $x' \in M$ with $v(x') > K$ such that*

$$p_j(x') \leq \frac{1}{K} \quad \text{and} \quad |x_k + x'_k| \geq |f_k| \quad \text{for some } k > K.$$

Proof. Let $k_0 > K$ be such that

$$p_j\Big(\sum_{k=k_0}^{\infty} x_k e_k\Big) \leq \frac{1}{2K} \quad \text{and} \quad p_j\Big(\sum_{k=k_0}^{\infty} f_k e_k\Big) \leq \frac{1}{2K}. \tag{2.1}$$

By assumption, we know that for any $k \geq 0$, M contains a non-zero vector y with valuation $v(y)$ bigger than k. Since M is homogeneous, we can thus choose $x' \in M$ such that

$$v(x') \geq k_0 \quad \text{and} \quad p_j(x') = 1/K.$$

Using (2.1), we deduce that

$$p_j\Big(\sum_{k=k_0}^{\infty} (x_k + x'_k) e_k\Big) \geq p_j(x') - p_j\Big(\sum_{k=k_0}^{\infty} x_k e_k\Big) \geq \frac{1}{K} - \frac{1}{2K} \geq p_j\Big(\sum_{k=k_0}^{\infty} f_k e_k\Big),$$

and thus

$$\sum_{k=k_0}^{\infty} |x_k + x'_k|^p a_{j,k}^p \geq \sum_{k=k_0}^{\infty} |f_k|^p a_{j,k}^p.$$

We conclude that there exists $k \geq k_0$ such that $|x_k + x'_k| \geq |f_k|$. $\square$

Remark 2.4. Since it is obvious that an infinite-dimensional subspace of $\lambda^p(A^+)$ is homogeneous and contains non-zero vectors of arbitrary large valuations, in what follows we will use the previous lemma in the particular case where M is a closed infinite-dimensional subspace of $\lambda^p(A^+)$.
It is worth noting that with the assumption that M is a closed infinite-dimensional subspace of $\lambda^p(A^+)$, this lemma can also be proven using the Baire Category Theorem through the fact that a subset with non-empty interior of a closed infinite-dimensional subspace cannot be σ-compact.

Theorem 2.5. *Let B_w be a weighted shift on $\lambda^p(A^+)$. If there exist a sequence $(f_k)_{k\geq 0} \in \lambda^p(A^+)$ and $j_0 \geq 1$ such that*

$$\frac{\#\{n \leq k : p_{j_0}(B_w^n(f_k e_k)) \geq 1\}}{k} \xrightarrow[k\to\infty]{} 1, \tag{2.2}$$

then B_w does not possess any frequently hypercyclic subspace in $\lambda^p(A^+)$.

Proof. By Theorem 2.1, we know that it suffices to prove that for any infinite-dimensional closed subspace M of X, any $K \geq 1$, any $x \in X$, any $j \geq 1$, there exists $x' \in M$ such that

$$p_j(x') \leq \frac{1}{K} \quad \text{and} \quad \sup_{k>K} \frac{\#\{n \leq k : p_{j_0}(B_w^n(x + x')) \geq 1\}}{k} \geq 1 - \frac{1}{K}.$$

Let M be an infinite-dimensional closed subspace of X, $K \geq 1$, $x \in X$ and $j \geq 1$. By assumption, we know that there exists $K_0 \geq K$ such that for any $k > K_0$,

$$\frac{\#\{n \leq k : p_{j_0}(B_w^n(f_k e_k)) \geq 1\}}{k} \geq 1 - \frac{1}{K}$$

Moreover, we deduce from Lemma 2.3 that there exists $x' \in M$ such that,

$$p_j(x') \leq \frac{1}{K_0} \leq \frac{1}{K} \quad \text{and} \quad |x_{\mathfrak{K}} + x'_{\mathfrak{K}}| \geq |f_{\mathfrak{K}}| \quad \text{for some } \mathfrak{K} > K_0.$$

Therefore, we obtain the desired result since

$$\begin{aligned} \sup_{k>K} \frac{\#\{n \leq k : p_{j_0}(B_w^n(x + x')) \geq 1\}}{k} &\geq \frac{\#\{n \leq \mathfrak{K} : p_{j_0}(B_w^n(f_{\mathfrak{K}} e_{\mathfrak{K}})) \geq 1\}}{\mathfrak{K}} \\ &\geq 1 - \frac{1}{K}. \end{aligned}$$

$\square$

Theorem 2.5 can be applied to the differentiation operator D on $H(\mathbb{C})$. Indeed, the space $H(\mathbb{C})$ can be seen as the Köthe sequence space $\lambda^1(A^+)$ with $a_{j,k} = j^k$, where an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is identified with the sequence $(a_k)_{k \geq 0}$. Moreover, the differentiation operator D on $H(\mathbb{C})$ then acts as the weighted shift B_w with $w_k = k$. In order to prove that the differentiation operator D does not possess any frequently hypercyclic subspace, it thus suffices to find a sequence $f \in H(\mathbb{C})$ such that

$$\frac{\#\{n \leq k : p_1(D^n(f_k e_k)) \geq 1\}}{k} \xrightarrow[k \to \infty]{} 1,$$

where $p_1(x) = \sum_{k=0}^{\infty} |x_k|$. This is the purpose of the following lemma.

Lemma 2.6. *Let* $\varepsilon_k = (\sqrt{\log(k+2)})^{-1}$ *and* $b_k = k\varepsilon_k$. *The sequence* $(f_k)_{k \geq 0}$, *defined by* $f_k = \prod_{j=0}^{k} \varepsilon_j$ *belongs to* $H(\mathbb{C})$ *and for any* $\delta \in]0, 1]$, *there exists* $k_0 \geq 1$ *such that for any* $k \geq k_0$,

$$\{n \leq k : p_1(D^n(f_k e_k)) \geq 1\} \supset [\lfloor \delta b_k \rfloor, k].$$

Proof. We first remark that $(f_k)_{k\geq 0}\in H(\mathbb{C})$ since $\varepsilon_k\to 0$ as $k\to\infty$. On the other hand, if we consider $k\geq 1$ and $n\in[\lfloor\delta b_k\rfloor,k]$, we have

$$\begin{aligned}
&p_1(D^n(f_ke_k))\\
&\quad=\frac{k!}{(k-n)!}f_k\geq\frac{k!}{(k-\lfloor\delta b_k\rfloor)!}f_k\\
&\quad\geq(k-\delta b_k)^{\lfloor\delta b_k\rfloor}f_k\\
&\quad=\exp\left(\lfloor\delta b_k\rfloor\log(k-\delta b_k)+\sum_{j=0}^{k}\log(\varepsilon_j)\right)\\
&\quad=\exp\left(\lfloor\delta b_k\rfloor\log(k)+\lfloor\delta b_k\rfloor\log\left(1-\delta\frac{b_k}{k}\right)-\frac{1}{2}\sum_{j=0}^{k}\log(\log(j+2))\right)\\
&\quad\geq\exp\Bigg(\left\lfloor\frac{\delta k}{\sqrt{\log(k+2)}}\right\rfloor\log(k)+\left\lfloor\frac{\delta k}{\sqrt{\log(k+2)}}\right\rfloor\log\left(1-\frac{\delta}{\sqrt{\log(k+2)}}\right)\\
&\qquad\qquad-\frac{1}{2}(k+1)\log(\log(k+2))\Bigg).
\end{aligned}$$

We can deduce from these inequalities that

$$\inf_{n\in[\lfloor\delta b_k\rfloor,k]}p_1(D^n(f_ke_k))\xrightarrow[k\to\infty]{}+\infty$$

and we conclude that there exists $k_0\geq 1$ such that for any $k\geq k_0$,

$$\{n\leq k:p_1(D^n(f_ke_k))\geq 1\}\supset[\lfloor\delta b_k\rfloor,k].$$

□

Theorem 2.7. *Let D be the differentiation operator on $H(\mathbb{C})$. Then D does not possess any frequently hypercyclic subspace.*

Proof. Let $\varepsilon_k=(\sqrt{\log(k+2)})^{-1}$ and $b_k=k\varepsilon_k$. We consider the sequence $(f_k)_{k\geq 0}\in H(\mathbb{C})$ defined by $f_k=\prod_{j=0}^{k}\varepsilon_j$. By Lemma 2.6, we know that there exists $k_0\geq 1$ such that for any $k\geq k_0$,

$$\{n\leq k:p_1(D^n(f_ke_k))\geq 1\}\supset[\lfloor b_k\rfloor,k].$$

In particular, we have, for any $k\geq k_0$,

$$\frac{\#\{n\leq k:p_1(D^n(f_ke_k))\geq 1\}}{k}\geq\frac{k-b_k}{k}=1-\frac{1}{\sqrt{\log(k+2)}}\xrightarrow[k\to\infty]{}1.$$

We conclude thanks to Theorem 2.5. □

3. Case of operators $P(D)$

In order to prove that D does not possess a frequently hypercyclic subspace, we have used the fact that for any $n,k\geq 0$,

$$p_j(D^n(x+x'))\geq p_j(D^n((x_k+x'_k)e_k)).$$

Unfortunately, we cannot use such an inequality if we consider the operator $P(D)$ where P is a polynomial with degree greater than 2. Indeed, several coordinates can balance out each other. As this counterbalance depends mainly on the coefficients of the polynomial P^n, we start by proving the following lemma.

Lemma 3.1. *Let $P(X) = \sum_{l=0}^{d} a_l X^l$ be a polynomial. There exists a constant $C > 0$ such that for any $n \geq 0$, we have $P^n(X) = a_d^n X^{nd} + \sum_{l=1}^{nd} b_{l,n} X^{nd-l}$ with*

$$|b_{l,n}| \leq C^l \sum_{p=1}^{l} \binom{n}{p} |a_d|^{n-p},$$

where $\binom{n}{p} = 0$ if $p > n$.

Proof. Since $P^n(X) = X^{nd}(\sum_{l=0}^{d} a_l X^{l-d})^n = X^{nd}(\sum_{l=0}^{d} a_{d-l} X^{-l})^n$, we deduce that $b_{l,n}$ is the coefficient of X^l in $R^n(X)$ where $R(X) = \sum_{l=0}^{d} a_{d-l} X^l$. If we denote by $Q(X)$ the polynomial $\sum_{l=1}^{d} a_{d-l} X^l$, we get that

$$R^n(X) = (a_d + Q(X))^n = a_d^n + \sum_{p=1}^{n} \binom{n}{p} a_d^{n-p} Q^p(X).$$

Since we are interested in the coefficient of X^l in $R^n(X)$, we have to compute the coefficient of X^l in $Q^p(X)$. For any $p \geq 1$, we have

$$Q^p(X) = \sum_{l_1,\dots,l_p=1}^{d} a_{d-l_1} \cdots a_{d-l_p} X^{l_1+\dots+l_p}$$

and thus the coefficient of X^l in $Q^p(X)$ is bounded by $\#\{1,\dots,d\}^p a^p = d^p a^p$ where $a := \max\{|a_0|,\cdots,|a_{d-1}|\}$. Moreover, we remark that since $Q(X)$ has no constant term, the coefficient of X^l in $Q^p(X)$ is equal to 0 if $p > l$. In conclusion, for any $l \geq 1$, we have

$$|b_{l,n}| \leq \sum_{p=1}^{l} \binom{n}{p} |a_d|^{n-p} d^p a^p \leq C^l \sum_{p=1}^{l} \binom{n}{p} |a_d|^{n-p},$$

where $C = \max\{da, 1\}$ and $\binom{n}{p} = 0$ if $p > n$. □

By using the above lemma, Lemma 2.3 and the properties of the sequence $f \in H(\mathbb{C})$ given by Lemma 2.6, we can now prove that if P is a polynomial then $P(D)$ satisfies the assumptions of Theorem 2.1.

Theorem 3.2. *Let D be the differentiation operator on $H(\mathbb{C})$ and P be a polynomial. Then $P(D)$ does not possess any frequently hypercyclic subspace.*

Proof. We first remark that if $\deg P \leq 0$ then $P(D) = \lambda Id$ for some $\lambda \in \mathbb{C}$ and $P(D)$ is not hypercyclic. We can thus assume that $P(D) = \sum_{l=0}^{d} a_l D^l$ where $d \geq 1$ and $a_d \neq 0$.

Let M be an infinite-dimensional closed subspace in X, $K\geq 1$, $x\in X$ and $j\geq 1$. In view of Theorem 2.1, it suffices to show that there exists $x'\in M$ such that $p_j(x')\leq \frac{1}{K}$ and

$$\sup_{k>K}\frac{\#\{n\leq k: p_1(P^n(D)(x+x'))\geq \frac{1}{4}\}}{k}\geq 1-\frac{1}{K}. \tag{3.1}$$

The vector x' will be obtained by using Lemma 2.3 with a convenient sequence $\tilde{f}\in H(\mathbb{C})$. We begin the proof by building this sequence.

By Lemma 3.1, we know that there exists a constant $C>0$ such that for any $n\geq 0$, we have $P^n(X)=a_d^nX^{nd}+\sum_{l=1}^{nd}b_{l,n}X^{nd-l}$ with

$$|b_{l,n}|\leq C^l\sum_{p=1}^{l}\binom{n}{p}|a_d|^{n-p}. \tag{3.2}$$

Let $\varepsilon_k=\frac{1}{\sqrt{\log(k+2)}}$, $b_k=k\varepsilon_k$ and $f_k=\prod_{j=0}^k\varepsilon_j$. We consider the sequence $f_k^{(\lambda)}=\lambda^kf_k$ where

$$\lambda=a\max\left\{1,\frac{2C}{\log\frac{3}{2}},8C\right\}\ \text{ and } a=\max\left\{\frac{1}{|a_d|},1\right\}. \tag{3.3}$$

Since $(f_k)\in H(\mathbb{C})$, we know that $(f_k^{(\lambda)})\in H(\mathbb{C})$. Moreover, by Lemma 2.6, we know that there exists $k_0>K$ such that for any $k\geq k_0$,

$$\{n\leq k: p_1(D^n(f_ke_k))\geq 1\}\supset\left[\left\lfloor\frac{1}{Kd}b_k\right\rfloor,k\right]. \tag{3.4}$$

and we consider $k_1\geq k_0$ such that for any $k\geq k_1$, we have

$$\left\lfloor\frac{b_k}{d}\right\rfloor>K\quad\text{and}\quad\left\lfloor\frac{1}{Kd}b_k\right\rfloor\geq 3. \tag{3.5}$$

We finally set $\tilde{f}:=C_1f^{(\lambda)}$ where C_1 is a real number bigger than 1 such that

$$|x_k|<\tilde{f}_k\quad\text{for any } k\leq k_1 \tag{3.6}$$

By using Lemma 2.3, we then obtain a vector $x'\in M$ with $v(x')>k_1$ such that,

$$p_j(x')\leq\frac{1}{K}\quad\text{and}\quad|x_k+x'_k|\geq\tilde{f}_k\quad\text{for some } k>k_1.$$

Let $\mathfrak{K}:=\min\{k\geq 0:|x_k+x'_k|\geq\tilde{f}_k\}$. Since $v(x')>k_1$, we deduce from (3.6) that $\mathfrak{K}>k_1$. It therefore suffices to prove that inequality (3.1) is satisfied for $k=\lfloor\frac{b_{\mathfrak{K}}}{d}\rfloor$ since $\mathfrak{K}>k_1$ and thus $\lfloor\frac{b_{\mathfrak{K}}}{d}\rfloor>K$ by (3.5).

Let $n\leq\frac{\mathfrak{K}}{d}$ and $y=P^n(D)(x+x')=(a_d^nD^{nd}+\sum_{l=1}^{nd}b_{l,n}D^{nd-l})(x+x')$. We obtain by definition of $\mathfrak{K}$ and of $\tilde{f}$

$$\begin{aligned}
&p_1(P^n(D)(x+x')) \\
&\quad\geq |y_{\mathfrak{K}-nd}| = \left| a_d^n D^{nd}((x_{\mathfrak{K}}+x'_{\mathfrak{K}})e_{\mathfrak{K}}) + \sum_{l=1}^{nd} b_{l,n} D^{nd-l}((x_{\mathfrak{K}-l}+x'_{\mathfrak{K}-l})e_{\mathfrak{K}-l}) \right| \\
&\quad\geq |a_d|^n D^{nd}(\tilde{f}_{\mathfrak{K}} e_{\mathfrak{K}}) - \sum_{l=1}^{nd} |b_{l,n}| D^{nd-l}(\tilde{f}_{\mathfrak{K}-l} e_{\mathfrak{K}-l}) \quad \text{by definition of } \mathfrak{K} \\
&\quad= C_1 |a_d|^n D^{nd}(f_{\mathfrak{K}}^{(\lambda)} e_{\mathfrak{K}}) - C_1 \sum_{l=1}^{nd} |b_{l,n}| D^{nd-l}(f_{\mathfrak{K}-l}^{(\lambda)} e_{\mathfrak{K}-l}) \\
&\quad= C_1 |a_d|^n \lambda^{\mathfrak{K}} \frac{\mathfrak{K}!}{(\mathfrak{K}-nd)!} f_{\mathfrak{K}} \left(1 - \sum_{l=1}^{nd} \frac{1}{\lambda^l} \frac{|b_{l,n}|}{|a_d|^n} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} \right) \\
&\quad\geq |a_d|^n \lambda^{\mathfrak{K}} \frac{\mathfrak{K}!}{(\mathfrak{K}-nd)!} f_{\mathfrak{K}} \left(1 - \sum_{l=1}^{nd} \frac{1}{\lambda^l} \frac{|b_{l,n}|}{|a_d|^n} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} \right) \quad \text{since } C_1 \geq 1.
\end{aligned}$$

Moreover, thanks to (3.2) and (3.3), we know that

$$\begin{aligned}
\sum_{l=1}^{nd} \frac{1}{\lambda^l} \frac{|b_{l,n}|}{|a_d|^n} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} &\leq \sum_{l=1}^{nd} \left(\frac{C}{\lambda}\right)^l \sum_{p=1}^{l} \binom{n}{p} |a_d|^{-p} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} \\
&\leq \sum_{l=1}^{nd} \left(\frac{aC}{\lambda}\right)^l \sum_{p=1}^{l} \binom{n}{p} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}}.
\end{aligned}$$

We divide the sum over l into two parts. On the one hand, when $l \leq \lceil \frac{n}{2} \rceil$ and $p \leq l$, then $\binom{n}{p} \leq \binom{n}{l}$ so that

$$\begin{aligned}
&\sum_{l=1}^{\lceil \frac{n}{2} \rceil} \left(\frac{aC}{\lambda}\right)^l \sum_{p=1}^{l} \binom{n}{p} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} \\
&\qquad\leq \sum_{l=1}^{\lceil \frac{n}{2} \rceil} \left(\frac{aC}{\lambda}\right)^l l \binom{n}{l} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} \\
&\qquad\leq \sum_{l=1}^{\lceil \frac{n}{2} \rceil} \left(\frac{2aC}{\lambda}\right)^l \binom{n}{l} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} \\
&\qquad\leq \sum_{l=1}^{\lceil \frac{n}{2} \rceil} \frac{1}{l!} \left(\frac{2aC}{\lambda}\right)^l \frac{n \cdots (n-l+1)}{\mathfrak{K} \cdots (\mathfrak{K}-l+1) \varepsilon_{\mathfrak{K}} \cdots \varepsilon_{\mathfrak{K}-l+1}}.
\end{aligned}$$

Let $u_l := \frac{n \cdots (n-l+1)}{\mathfrak{K} \cdots (\mathfrak{K}-l+1) \varepsilon_{\mathfrak{K}} \cdots \varepsilon_{\mathfrak{K}-l+1}}$ for any $1 \leq l \leq n$. We remark that for any $n \leq b_{\mathfrak{K}}$

$$u_1 = \frac{n}{\mathfrak{K}\varepsilon_{\mathfrak{K}}} \leq \frac{b_{\mathfrak{K}}}{\mathfrak{K}\varepsilon_{\mathfrak{K}}} = 1.$$

Moreover, for any $n \leq b_{\mathfrak{K}}$, any $1 \leq l \leq n$

$$\begin{aligned}\frac{u_{l+1}}{u_l} = \frac{n-l}{(\mathfrak{K}-l)\varepsilon_{\mathfrak{K}-l}} &\leq \frac{(b_{\mathfrak{K}}-l)\sqrt{\log(\mathfrak{K}-l+2)}}{(\mathfrak{K}-l)} \\ &= \frac{(\mathfrak{K}-l\sqrt{\log(\mathfrak{K}+2)})}{(\mathfrak{K}-l)} \frac{\sqrt{\log(\mathfrak{K}-l+2)}}{\sqrt{\log(\mathfrak{K}+2)}} \leq 1.\end{aligned}$$

We conclude that $u_l \leq 1$ for any $1 \leq l \leq n$. This yields

$$\begin{aligned}\sum_{l=1}^{\lceil \frac{n}{2} \rceil} \left(\frac{aC}{\lambda}\right)^l \sum_{p=1}^{l} \binom{n}{p} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} &\leq \sum_{l=1}^{\lceil \frac{n}{2} \rceil} \frac{1}{l!} \left(\frac{2aC}{\lambda}\right)^l \\ &\leq e^{\frac{2aC}{\lambda}} - 1 \\ &\leq \frac{1}{2} \text{ by (3.3) .}\end{aligned}$$

On the other hand, it follows from (3.5) that if $n \geq \lfloor \frac{1}{Kd} b_{\mathfrak{K}} \rfloor$ then $\lceil \frac{n}{2} \rceil + 1 \geq 3$. Observing that for all $n \geq \lfloor \frac{1}{Kd} b_{\mathfrak{K}} \rfloor$ and $l \geq \lceil \frac{n}{2} \rceil + 1$

$$\sum_{p=1}^{l} \binom{n}{p} \leq 2^n \text{ and } \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} \leq 1$$

we deduce that

$$\begin{aligned}\sum_{l=\lceil \frac{n}{2} \rceil+1}^{nd} \left(\frac{aC}{\lambda}\right)^l \sum_{p=1}^{l} \binom{n}{p} \frac{(\mathfrak{K}-l)!}{\mathfrak{K}!} \frac{f_{\mathfrak{K}-l}}{f_{\mathfrak{K}}} &\leq \sum_{l=\lceil \frac{n}{2} \rceil+1}^{nd} \left(\frac{aC}{\lambda}\right)^l 2^n \\ &\leq \sum_{l=\lceil \frac{n}{2} \rceil+1}^{nd} \left(\frac{4aC}{\lambda}\right)^l \\ &\leq \sum_{l=3}^{+\infty} \frac{1}{2^l} \text{ by (3.3)} \\ &\leq \frac{1}{4}.\end{aligned}$$

Summarizing the previous estimates, we get that for all $n \in [\lfloor \frac{1}{Kd} b_{\mathfrak{K}} \rfloor, \frac{b_{\mathfrak{K}}}{d}]$, we have

$$\begin{aligned}p_1(P^n(D)(x+x')) &\geq |a_d|^n \lambda^{\mathfrak{K}} \frac{\mathfrak{K}!}{(\mathfrak{K}-nd)!} f_{\mathfrak{K}} \left(1 - \frac{1}{2} - \frac{1}{4}\right) \\ &\geq \frac{1}{4} \frac{\mathfrak{K}!}{(\mathfrak{K}-nd)!} f_{\mathfrak{K}} \quad \text{by (3.3)} \\ &= \frac{1}{4} p_1(D^{nd}(f_{\mathfrak{K}} e_{\mathfrak{K}})) \geq \frac{1}{4} \quad \text{by (3.4).}\end{aligned}$$

We deduce that

$$\begin{aligned}
&\sup_{k>K} \frac{\#\{n \leq k : p_1(P^n(D)(x+x')) \geq \frac{1}{4}\}}{k} \\
&\quad \geq \frac{\#\{n \leq \lfloor \frac{b_{\mathfrak{K}}}{d} \rfloor : p_1(P^n(D)(x+x')) \geq \frac{1}{4}\}}{\frac{b_{\mathfrak{K}}}{d}} \\
&\quad \geq \frac{\lfloor \frac{b_{\mathfrak{K}}}{d} \rfloor - \lfloor \frac{1}{Kd} b_{\mathfrak{K}} \rfloor + 1}{\frac{b_{\mathfrak{K}}}{d}} \\
&\quad \geq \frac{\frac{b_{\mathfrak{K}}}{d} - \frac{1}{Kd} b_{\mathfrak{K}}}{\frac{b_{\mathfrak{K}}}{d}} = 1 - \frac{1}{K}.
\end{aligned}$$

This concludes the proof. □

Since non-constant polynomials in the differentiation operator are frequently hypercyclic on $H(\mathbb{C})$, we can deduce from an argument of Godefroy and Shapiro [13] that these operators are also frequently hypercyclic on the space $\mathcal{C}^\infty(\mathbb{R})$. Thus the following question arises.

Question 4. Does the operator $P(D)$ have a (frequently) hypercyclic subspace on the space $\mathcal{C}^\infty(\mathbb{R})$ where P is a non-constant polynomial?

It seems that we cannot answer this question with the same method than on the space $H(\mathbb{C})$. Indeed, the differentiation operator on the space of entire functions acts like a shift operator and this allowed us to simplify greatly the conditions of Theorem 2.1 but this is not possible anymore on $\mathcal{C}^\infty(\mathbb{R})$. Moreover, the technique used in [13] does not permit to transfer the non-existence of frequently hypercyclic subspaces from $H(\mathbb{C})$ to $\mathcal{C}^\infty(\mathbb{R})$.

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